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NUMERICAL SOLUTIONS OF THE INTEGRO-DIFFERENTIAL EQUATIONS OF HIGH-SPEED RADIATING BOUNDARY LAYERS

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NOMENCLATURE

C_p, specific heat at constant pressure;
E_n, exponential integral function defined in equation (7);
e_b, Planck's radiation function;
H, stagnation enthalphy;
l, mixing length;
Pr_{eff}, effective Prandtl number;
q_R, radiant heat flux vector;
Re_x, local Reynolds number;
T, temperature;
u, velocity, x-direction;
v, velocity, y-direction;
x, length, parallel to body surface;
y, length, normal to body surface;
δ, boundary layer thickness;
κ, absorption coefficient;
μ, laminar viscosity;
μ_{eff}, effective viscosity;
ρ, density;
τ, shear stress;
τ_λ, optical coordinate defined in equation (5);
$\tau_{0\lambda}$, optical thickness of boundary layer defined in equation (6);
Φ, generation term for ϕ [5].

Subscripts

D, downstream point in the finite difference grid;
R, radiation term;
U, upstream point in the finite difference grid;
w, body surface condition;
λ, wavelength;
0, reference condition;
∞, freestream condition.

Bold symbols indicate a vector quantity.

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INTRODUCTION

DUE TO the integro-differential nature of the governing equations for radiating boundary layer flow, the optically thin and optically thick approximations are usually applied to the boundary layer. However, analyses of radiating boundary layers which include the integral radiation transport terms take various approaches to the problem. Viskanta and Grosh [1] transform the integro-differential energy equation into a non-linear integral equation and then obtain a solution by successive approximations. Hoshizaki and Wilson [2] employ a combined finite-difference-integral method to analyze viscous flow about a blunt, high-speed body including radiation transport, with the energy and momentum equations solved in an uncoupled manner. Another approach to this type problem is used by Shen [3] who approximates the integral radiation terms with differential terms; the highest order derivative in the resulting energy equation is the same order as that for the simple conduction term.

In this analysis the integral terms are retained and a modified Patankar–Spalding finite-difference procedure [4] is employed to obtain solutions to the governing equations of the turbulent, radiating boundary layer. This represents an extension of the authors' previous work with the Patankar–Spalding technique in optically thin flows [5].

Steady, turbulent flow past a black, flat plate with constant surface temperature at zero angle of attack is to be analyzed including the effects of compressibility and viscous dissipation. The effect of radiation transport parallel to the plate is neglected.

ANALYSIS

The governing equations to be solved by finite-difference procedure are given by

Continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

x-Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu_{\text{eff}} \frac{\partial u}{\partial y}\right) \tag{2}$$

Energy

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{\mu_{\text{eff}}}{Pr_{\text{eff}}} \frac{\partial H}{\partial y} + \mu_{\text{eff}}\left(1 - \frac{1}{Pr_{\text{eff}}}\right)\frac{1}{2}\frac{\partial u^2}{\partial y}\right\} - \operatorname{div} q_R \tag{3}$$

Boundary conditions

$$\text{at } y = 0, u = v = 0, H = H_w$$

$$\text{as } y \to \infty, u \to u_\infty, H \to H_\infty.$$

In equations (1)–(3), u, v and H represent the time averaged values of the fluctuating turbulent quantities.

Assuming that the flat plate surface and freestream emit energy into the boundary layer as black radiators at their respective temperatures, the divergence of the radiative heat flux vector may be written as

$$-\operatorname{div} \boldsymbol{q}_R = \int_0^\infty 2\kappa_\lambda \Big\{e_{bw\lambda} E_2(\tau_\lambda) + e_{b\infty\lambda} E_2(\tau_{0\lambda} - \tau_\lambda) + \int_0^{\tau_{0\lambda}} e_{b\lambda}(t) E_1(|\tau_\lambda - t|)\, dt - 2e_{b\lambda}(\tau_\lambda)\Big\}\, d_\lambda \tag{4}$$

where τ_λ is an optical coordinate defined by

$$\tau_\lambda = \int_0^y \kappa_\lambda \, dy'. \tag{5}$$

$\tau_{0\lambda}$ is the optical thickness of the boundary layer defined by

$$\tau_{0\lambda} = \int_0^\delta \kappa_\lambda \, dy' \tag{6}$$

and $E_n(t)$ are exponential integrals defined by

$$E_n(t) = \int_0^1 s^{n-2}\, e^{-t/s}\, ds. \tag{7}$$

Employing a band approximation for the absorption coefficient on N wavelength intervals, the absorption coefficient may be expressed as

$$\kappa_\lambda = \begin{cases} \kappa_1 & 0 \leqslant \lambda < \lambda_2 \\ \kappa_2 & \lambda_2 \leqslant \lambda < \lambda_3 \\ \vdots & \vdots \\ \kappa_N & \lambda_N \leqslant \lambda < \infty \end{cases} \tag{8}$$

where $\lambda_1 \equiv 0$ and $\kappa_1 \ldots \kappa_N$ are constant with wavelength over their respective band.

Equation (4) may now be re-written in terms of the band model absorption coefficient

$$-\operatorname{div} \boldsymbol{q}_R = \sum_{i=1}^N 2\kappa_i \Big\{e_{bwi} E_2(\tau_i) + e_{b\infty i} E_2(\tau_{0i} - \tau_i) + \int_0^{\tau_{0i}} e_{bi}(t) E_1(|\tau_i - t|)\, dt - 2e_{bi}(\tau_i)\Big\} \tag{9}$$

where

$$\tau_i = \int_0^y \kappa_i \, dy' \tag{10}$$

and

$$e_{bi} = \int_{\lambda_i}^{\lambda_{i+1}} \frac{C_1}{\lambda^5 (e^{C_2/\lambda T} - 1)}\, d\lambda, \tag{11}$$

C_1 and C_2 bring well-known constants.

As shown by the authors [5]

$$\Phi_R = -\operatorname{div} \boldsymbol{q}_R \tag{12}$$

where Φ_R may be thought of as a radiation source term for the energy equation. To make equation (9) compatible with the solution technique, it must be written, in terms of Φ_R, as

$$\left.\frac{\Phi_R}{\rho u}\right|_D = \left.\frac{\Phi_R}{\rho u}\right|_U + \left.\frac{\partial}{\rho H}\left(\frac{\Phi_R}{\rho u}\right)\right|_U (H_D - H_U) \tag{13}$$

relating the unknown value of the source term at a downstream finite-difference grid point to a known value and its derivative upstream and the value of the dependent variable H at both positions. Assuming that air behaves as a perfect gas, it can be shown that

$$\left.\frac{\Phi_R}{\rho u}\right|_D = -\frac{1}{\rho u}\,\mathrm{div}\,\left.\mathbf{q}_R\right|_D = -\frac{1}{\rho u}\,\mathrm{div}\,\left.\mathbf{q}_R\right|_U - \frac{1}{\rho u C_p}\left[\frac{\partial}{\partial T}(\mathrm{div}\,\mathbf{q}_R) + \frac{1}{T}\mathrm{div}\,\mathbf{q}_R\right]_U (H_D - H_U) \tag{14}$$

which is included in the general difference equation.

RESULTS

To investigate the effects of non-gray radiation transport, the three band absorption coefficient suggested by Smith and Hassan [6] is employed:

$$\kappa_1 = 0 \qquad 0 \leqslant \lambda < 0{\cdot}05\,\mu$$

$$\kappa_2 = 437\left(\frac{\rho}{\rho_0}\right)^{1{\cdot}009}\left(\frac{T}{10^4}\right)^{2{\cdot}85}\ \mathrm{m}^{-1}$$

$$0{\cdot}05\,\mu \leqslant \lambda < 0{\cdot}135\,\mu$$

$$\kappa_3 = 4{\cdot}985\left(\frac{\rho}{\rho_0}\right)^{1{\cdot}205}\left(\frac{T}{10^4}\right)^{5{\cdot}47}\ \mathrm{m}^{-1}$$

$$0{\cdot}135\,\mu \leqslant \lambda. \tag{15}$$

Following Patankar and Spalding, the effective viscosity μ_{eff} is evaluated from Prandtl's mixing length hypothesis together with a modified Van Driest damping theory. The laminar viscosity is specified by

$$\mu = \mu_0\left(\frac{T}{T_0}\right)^{0{\cdot}76}. \tag{16}$$

The freestream and plate surface temperatures are assumed constant, both at 296°K and the freestream pressure is assumed to be $1{\cdot}013 \times 10^5$ Ns/m^2. For the given boundary conditions, separate flows at Mach 1–5 are solved once neglecting radiation transport and once considering it. For each problem, the computational procedure begins at the x-station which corresponds to

$$Re_x = 3{\cdot}2 \times 10^5 \tag{17}$$

or at a point approximately where the flow becomes fully turbulent. At the starting location, one-seventh power law velocity and enthalpy profiles are specified as initial conditions.

Figure 1 shows the dimensionless temperature profiles for the problems. From each set of profiles, it is evident that the maximum temperature in the boundary layer for the radiating case is lower than the maximum for the non-radiating flow. This is, of course, a result of energy loss due to radiation.

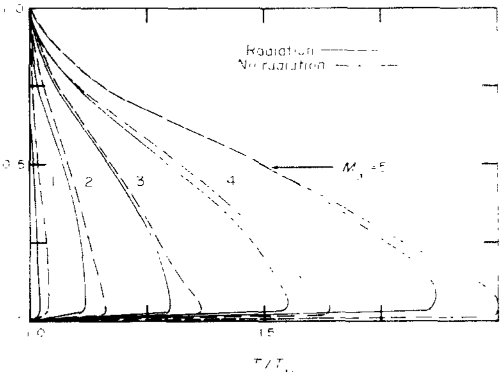

FIG. 1. Dimensionless temperature profiles at $Re_x = 1{\cdot}5 \times 10^6$.

Table 1 indicates a large decrease in surface convection at Mach 1 (51·38 per cent) from the non-radiating to the radiating case. The decrease is reduced to 18·0 per cent at Mach 5. The large decrease at the lower Mach numbers is thought to be as much a result of the rather coarse three band absorption coefficient model as the radiation loss. The radiant heat flux to the plate was found to be negligible for all cases with the given boundary conditions. Therefore, the convection results in Table 1 may be considered the total values.

Whereas the thermal phenomena in these flows are to be expected, some rather interesting results appear with regard to the turbulent nature of the problem. From the combined mixing length and modified Van Driest theories:

$$\mu_{\mathrm{eff}} = \rho l^2 \left|\frac{\partial u}{\partial y}\right| \tag{18}$$

Table 1. Decrease in surface convection as a function of Mach number at Re$_x$ = $1{\cdot}5 \times 10^6$

M_∞	q_w (non-radiating) Btu/ft^{2}s	q_w (radiating) Btu/ft^{2}s	% decrease
1	0·7988	0·3883	51·38
2	6·6167	5·0181	24·16
3	23·322	18·633	20·1
4	60·339	49·235	18·4
5	128·83	105·63	18·0

where

$$l = Ky\{1{\cdot}0 - \exp(-y\sqrt{\tau\rho}/\mu A_+)\}, \tag{19}$$

K and A_+ being constants. At a specific y location in the boundary layer, noting that

$$\tau \sim T^{0{\cdot}76} \tag{20}$$

since

$$\tau = \mu\frac{\partial u}{\partial y}\,(\text{laminar}) \tag{21}$$

and

$$\mu \sim T^{0{\cdot}6} \tag{22}$$

and

$$\rho \sim T^{-1{\cdot}0} \tag{23}$$

it can be shown that

$$l \sim K_1\{1{\cdot}0 - \exp(-K_2T^{-0{\cdot}88})\} \tag{24}$$

where K_1 and K_2 are constants. Therefore, if the temperature at a point in the boundary decreases, as is the case when radiation transport effects are considered, l increases and from equation (18) it is clear that μ_{eff} should also increase since ρ increases with decreasing temperature and the velocity profiles are essentially identical for the two different flows. By accounting for radiation transport then the local predicted temperatures are lower but the local shear stresses in the turbulent flow are higher.

SUMMARY AND CONCLUSIONS

The principal intent of this note is to demonstrate that the spectrally dependent, integral radiation transport equations can be included in the governing equations of the turbulent boundary layer and that solutions are possible through a modified Patankar–Spalding finite-difference procedure. Obviously, refinements should be made in the band model absorption coefficient. However, this analysis is not dependent on one particular band model or the number of bands desired. The results reported herein do agree qualitatively with previous radiating flow analyses with regard to the heat transfer. It is also shown that the predicted values for local shear stress are higher when radiation transport effects are accounted for.

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